ECE 604, Lecture 23

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1 Scalar and Vector Potentials for Time-Harmonic Fields

1.1 Introduction

Previously, we have studied the use of scalar potential Φ for electrostatic problems. Then we learnt the use of vector potential $\mathbf{A}$ for magnetostatic problems. Now, we will study the combined use of scalar and vector potential for solving time-harmonic (electrodynamic) fields.

This is important for bridging the gap between static regime where the frequency is zero or low, and dynamic regime where the frequency is not low. For the dynamic regime, it is important to understand the radiation of electromagnetic fields. Electrodynamic regime is important for studying antennas, communications, sensing, wireless power transfer applications, and many more. Hence, it is imperative to understand how time-varying electromagnetic fields radiate from sources.

It is also important to understand when static or circuit (quasi-static) regimes are important. The circuit regime solves problems that have fueled the microchip industry, and it is hence imperative to understand when electromagnetic problems can be approximated with simple circuit problems and solved using simple laws such as KCL an KVL.

1.2 Scalar and Vector Potentials for Statics, A Review

Previously, we have studied scalar and vector potentials for electrostatics and magnetostatics where the frequency ω is identically zero. The four Maxwell's equations for a homogeneous medium are then

$$\nabla \times \mathbf{E} = 0 \tag{1.1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.2}$$

$$\nabla \cdot \varepsilon \mathbf{E} = \varrho \tag{1.3}$$

$$\nabla \cdot \mu \mathbf{H} = 0 \tag{1.4}$$

In order to satisfy the first of Maxwell's equations or Faraday's law above, we let

$$\mathbf{E} = -\nabla \Phi \tag{1.5}$$

Using the above in (1.3), we get, for a homogeneous medium, that

$$\nabla^2 \Phi = -\frac{\varrho}{\varepsilon} \tag{1.6}$$

which is the Poisson's equation for electrostatics.

By letting

$$\mu \mathbf{H} = \nabla \times \mathbf{A} \tag{1.7}$$

since $\nabla\cdot(\nabla\times\mathbf{A}) = 0$, the last of Maxwell's equations above will be automatically satisfied. And using the above in the second of Maxwell's equations above, we get

$$\nabla \times \nabla \times \mathbf{A} = \mu\mathbf{J} \tag{1.8}$$

Now, using the fact that $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, and Coulomb's gauge that $\nabla \cdot \mathbf{A} = 0$, we arrive at

$$\nabla^2\mathbf{A} = -\mu\mathbf{J} \tag{1.9}$$

which is the vector Poisson's equation. Next, we will repeat the above derivation when $\omega \neq 0$.

1.3 Scalar and Vector Potentials for Electrodynamics

To this end, we will start with frequency domain Maxwell's equations with sources $\mathbf{J}$ and ϱ included, and later see how these sources $\mathbf{J}$ and ϱ can radiate electromagnetic fields. Maxwell's equations in the frequency domain are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{1.10}$$
$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \tag{1.11}$$
$$\nabla \cdot \mu\mathbf{H} = 0 \tag{1.12}$$
$$\nabla \cdot \varepsilon\mathbf{E} = \varrho \tag{1.13}$$

In order to satisfy the third Maxwell's equation, as before, we let

$$\mu\mathbf{H} = \nabla \times \mathbf{A} \tag{1.14}$$

Now, using (1.14) in (1.10), we have

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0 \tag{1.15}$$

Since $\nabla \times (\nabla\Phi) = 0$, the above implies that

$$\mathbf{E} = -\nabla\Phi - j\omega\mathbf{A} \tag{1.16}$$

The above implies that the electrostatic theory of $\mathbf{E} = -\nabla\Phi$ is not exactly correct when $\omega \neq 0$. The second term above, in accordance to Faraday's law, is the contribution to the electric field from the time-varying magnetic field, and hence, is the induction term.

Furthermore, the above shows that given $\mathbf{A}$ and Φ, one can determine the fields $\mathbf{H}$ and $\mathbf{E}$. To this end, we will derive equations for $\mathbf{A}$ and Φ in terms of the sources $\mathbf{J}$ and ϱ which are given. Substituting (1.14) and (1.16) into (1.11) gives

$$\nabla \times \nabla \times \mathbf{A} = -j\omega\mu\varepsilon(-j\omega\mathbf{A} - \nabla\Phi) + \mu\mathbf{J} \tag{1.17}$$

Or upon rearrangement, after using that $\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla \cdot \nabla \mathbf{A}$, we have

$$\nabla^2 \mathbf{A} + \omega^2 \mu\varepsilon \mathbf{A} = -\mu \mathbf{J} + j\omega\mu\varepsilon \nabla\Phi + \nabla\nabla \cdot \mathbf{A} \tag{1.18}$$

Moreover, using (1.16) in (1.13), we have

$$\nabla \cdot (j\omega \mathbf{A} + \nabla\Phi) = -\frac{\varrho}{\varepsilon} \tag{1.19}$$

In the above, (1.18) and (1.19) represent two equations for the two unknowns $\mathbf{A}$ and Φ, expressed in terms of the known quantities, the sources $\mathbf{J}$ and ϱ which are given. But these equations are coupled to each other. They look complicated and are rather difficult to solve at this point.

As in the magnetostatic case, the vector potential $\mathbf{A}$ is not unique. One can always construct a new $\mathbf{A}' = \mathbf{A} + \nabla\Psi$ that produces the same magnetic field $\mu\mathbf{H}$, since $\nabla \times \nabla\Psi = 0$. It is quite clear that $\mu\mathbf{H} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}'$. This implies that $\mathbf{A}$ is not unique, and one can further show that Φ is also non-unique.

To make them unique, in addition to specifying what $\nabla \times \mathbf{A}$ should be in (1.14), we need to specify its divergence or $\nabla \cdot \mathbf{A}$ as in the electrostatic case.[1]

A clever way to specify the divergence of $\mathbf{A}$ is to make it simplify the complicated equations above in (1.18). We choose a gauge so that the last two terms in the equation will cancel each other. Therefore, we specify

$$\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi \tag{1.20}$$

The above is judiciously chosen so that the pertinent equations (1.18) and (1.19) will be simplified and decoupled. Then they become

$$\nabla^2 \mathbf{A} + \omega^2 \mu\varepsilon \mathbf{A} = -\mu \mathbf{J} \tag{1.21}$$

$$\nabla^2 \Phi + \omega^2 \mu\varepsilon \Phi = -\frac{\varrho}{\varepsilon} \tag{1.22}$$

Equation (1.20) is known as the Lorenz gauge[2] and the above equations are Helmholtz equations with source terms. Not only are these equations simplified, they can be solved independently of each other since they are decoupled from each other.

Equations (1.21) and (1.22) can be solved using the Green's function method. Equation (1.21) actually implies three scalar equations for the three x, y, z components, namely that

$$\nabla^2 A_i + \omega^2 \mu\varepsilon A_i = -\mu J_i \tag{1.23}$$

[1] This is akin to that given a vector $\mathbf{A}$, and an arbitrary vector $\mathbf{k}$, in addition to specifying what $\mathbf{k} \times \mathbf{A}$ is, it is also necessary to specify what $\mathbf{k} \cdot \mathbf{A}$ is to uniquely specify $\mathbf{A}$.

[2] Please note that this Lorenz is not the same as Lorentz.

where i above can be x, y, or z. Therefore, (1.21) and (1.22) together constitute four scalar equations similar to each other. Hence, we need only to solve their point-source response, or the Green's function of these equations by solving

$$\nabla^2 g(\mathbf{r}, \mathbf{r}') + \beta^2 g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{1.24}$$

where $\beta^2 = \omega^2 \mu \varepsilon$.

Previously, we have shown that when $\beta = 0$,

$$g(\mathbf{r}, \mathbf{r}') = g(|\mathbf{r} - \mathbf{r}'|) = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

When $\beta \neq 0$, the correct solution is

$$g(\mathbf{r}, \mathbf{r}') = g(|\mathbf{r} - \mathbf{r}'|) = \frac{e^{-j\beta|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{1.25}$$

which can be verified by back substitution.

By using the principle of linear superposition, or convolution, the solutions to (1.21) and (1.22) are then

$$\mathbf{A}(\mathbf{r}) = \mu \iiint d\mathbf{r}' \mathbf{J}(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{1.26}$$

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon} \iiint d\mathbf{r}' \varrho(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{1.27}$$

In the above $d\mathbf{r}'$ is the shorthand notation for $dx\,dy\,dz$ and hence, they are still volume integrals.

2 When is Static Theory Valid?

We have learnt in the previous section that for electrodynamics,

$$\mathbf{E} = -\nabla\Phi - j\omega\mathbf{A} \tag{2.1}$$

where the second term above on the right-hand side is due to induction, or the contribution to the electric field from the time-varying magnetic filed. Hence, much things we learn in potential theory that $\mathbf{E} = -\nabla\Phi$ is not truly valid. But simple potential theory that $\mathbf{E} = -\nabla\Phi$ is very useful because of its simplicity. We will study when static electromagnetic theory can be used to model this world. Since the third and the fourth Maxwell's equations are derivable from the first two, let us first study when we can ignore the time derivative terms in the first two of Maxwell's equations, which, in the frequency domain, are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{2.2}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \tag{2.3}$$

When the terms multiplied by $j\omega$ above can be ignored, then electrodynamics can be replaced with static electromagnetics, which are much simpler. That is why Ampere's law, Coulomb's law, and Gauss' law were discovered first. Quasi-static electromagnetic theory eventually gave rise to circuit theory and telegraphy technology. Circuit theory consists of elements like resistors, capacitors, and inductors. Given that we have now seen electromagnetic theory in its full form, we like to ponder when we can use simple static electromagnetics to describe electromagnetic phenomena.

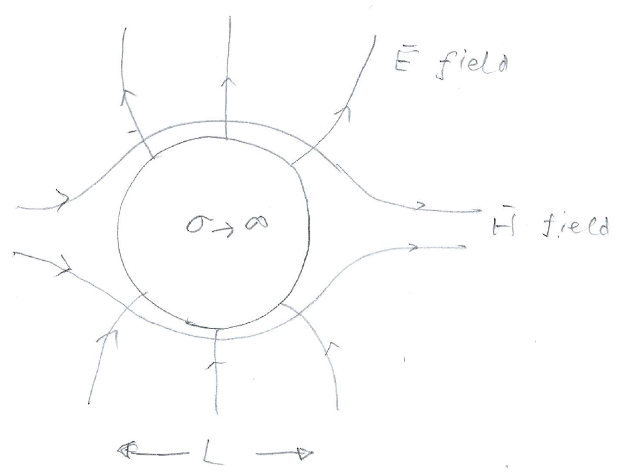

Figure 1: The electric and magnetic fields around a conducting particle contorts themselves to satisfy the boundary conditions even when the particle is very small.

To see this lucidly, it is best to write Maxwell's equations in dimensionless units or the same units. Say if we want to solve Maxwell's equations for the fields close to an object of size L as shown in Figure 1. This object can be a small particle like the sphere, or it could be a capacitor, or an inductor, which are small; but how small should it be before we can apply static electromagnetics?

It is clear that these $\mathbf{E}$ and $\mathbf{H}$ fields will have to satisfy boundary conditions, which is *de rigueur* in the vicinity of the object as shown in Figure 1 even when the frequency is low or the wavelength long. The fields become great contortionist in order to do so. Hence, we do not expect a constant field around the object but that the field will vary on the length scale of L. So we renormalize our length scale by this length L by defining a new dimensionless coordinate system such that.

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{L} \tag{2.4}$$

In other words, by so doing, then $Ldx' = dx$, $Ldy' = dy$, and $Ldz' = dz$, and

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{1}{L}\frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{1}{L}\frac{\partial}{\partial z'} \tag{2.5}$$

In this manner, $\nabla = \frac{1}{L}\nabla'$; or ∇ will be very large when it operates on fields that vary on the length scale of L, where ∇' will not be large because it is in coordinates normalized with respect to L.

Then, the first two of Maxwell's equations become

$$\frac{1}{L}\nabla' \times \mathbf{E} = -j\omega\mu_0\mathbf{H} \tag{2.6}$$

$$\frac{1}{L}\nabla' \times \mathbf{H} = j\omega\varepsilon_0\mathbf{E} + \mathbf{J} \tag{2.7}$$

Here, we still have apples and oranges to compare with since **E** and **H** have different units; we cannot compare quantities if they have different units. For instance, the ratio of **E** to the **H** field has a dimension of impedance. To bring them to the same unit, we define a new $\mathbf{E}'$ such that

$$\eta_0\mathbf{E}' = \mathbf{E} \tag{2.8}$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \cong 377$ ohms in vacuum. In this manner, the new $\mathbf{E}'$ has the same unit as the **H** field. Then, (2.6) and (2.7) become

$$\frac{\eta_0}{L}\nabla' \times \mathbf{E}' = -j\omega\mu_0\mathbf{H} \tag{2.9}$$

$$\frac{1}{L}\nabla' \times \mathbf{H} = j\omega\varepsilon_0\eta_0\mathbf{E}' + \mathbf{J} \tag{2.10}$$

With this change, the above can be rearranged to become

$$\nabla' \times \mathbf{E}' = -j\omega\mu_0\frac{L}{\eta_0}\mathbf{H} \tag{2.11}$$

$$\nabla' \times \mathbf{H} = j\omega\varepsilon_0\eta_0 L\mathbf{E}' + L\mathbf{J} \tag{2.12}$$

By letting $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$, the above can be further simplified to become

$$\nabla' \times \mathbf{E}' = -j\frac{\omega}{c_0}L\mathbf{H} \tag{2.13}$$

$$\nabla' \times \mathbf{H} = j\frac{\omega}{c_0}L\mathbf{E}' + L\mathbf{J} \tag{2.14}$$

Notice now that in the above, **H**, $\mathbf{E}'$, and $L\mathbf{J}$ have the same unit, and ∇' is dimensionless and is of order one, and $\omega L/c_0$ is also dimensionless.

Therefore, one can compare terms, and ignore the frequency dependent $j\omega$ term when

$$\frac{\omega}{c_0}L \ll 1 \tag{2.15}$$

or when

$$2\pi \frac{L}{\lambda_0} \ll 1 \tag{2.16}$$

Consequently, the above criteria are for the validity of the static approximation when the time-derivative terms in Maxwell's equations can be ignored. When these criteria are satisfied, then Maxwell's equations can be simplified to and approximated by the following equations

$$\nabla' \times \mathbf{E}' = 0 \tag{2.17}$$

$$\nabla' \times \mathbf{H} = L\mathbf{J} \tag{2.18}$$

which are the static equations, Faraday's law and Ampere's law of electromagnetic theory. They can be solved together with Gauss' laws.

In other words, one can solve, even in optics, where ω is humongous or the wavelength very short, using static analysis. This is illustrated in the field of nano-optics with a plasmonic nanoparticle. If the particle is small enough compared to wavelength of the light, electrostatic analysis can be used. For instance, the wavelength of blue light is about 400 nm, and 10 nm nano-particles can be made. (Even the ancient Romans could make them!) And hence, static electromagnetic theory can be used to analyze the wave-particle interaction. This was done in one of the homeworks. Figure 2 shows an incident light whose wavelength is much longer than the size of the particle. The incident field induces an electric dipole moment on the particle, whose external field can be written as

$$\mathbf{E}_s = (\hat{r} 2\cos\theta + \hat{\theta}\sin\theta)\left(\frac{a}{r}\right)^3 E_s \tag{2.19}$$

while the incident field and the interior field to the particle can be expressed as

$$\mathbf{E}_0 = \hat{z}E_0 = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)E_0 \tag{2.20}$$

$$\mathbf{E}_i = \hat{z}E_i = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)E_i \tag{2.21}$$

By matching boundary conditions, as was done in the homework, it can be shown that

$$E_s = \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} E_0 \tag{2.22}$$

$$E_i = \frac{3\varepsilon}{\varepsilon_s + 2\varepsilon} E_0 \tag{2.23}$$

For a plasmonic nano-particle, the particle medium behaves like a plasma, and ε_s in the above can be negative, making the denominators of the above expression close to zero. Therefore, the amplitude of the internal and scattered fields can be very large when this happens, and the nano-particles will glitter in the presence of light.

Figure 3 shows a nano-particle sets in plasmonic oscillation by a light wave. Figure 4 shows that different color fluids can be obtained by immersing nano-particles in fluids with different background permittivity causing the plasmonic particles to resonate at different frequencies.

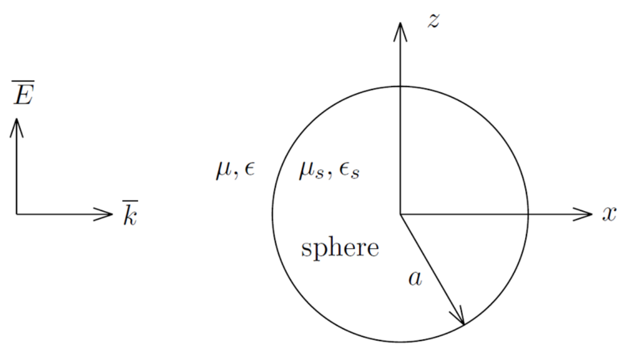

Figure 2: Courtesy of Kong.

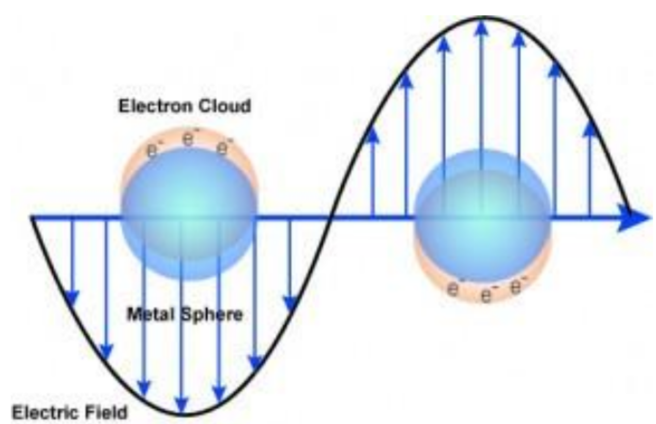

Figure 3: A nano-particle undergoing electromagnetic oscillation when an electromagnetic wave impinges on it.

Figure 4: Color of fluid containing nano-particles can be obtained by changing the permittivity of the background fluid.

In (2.16), this criterion has been expressed in terms of the dimension of the object L compared to the wavelength λ_0. Alternatively, we can express this criterion in terms of transit time. The transit time for an electromagnetic wave to traverse an object of size L is $\tau = L/c_0$ and $\omega = 2\pi/T$ where T is the period of the time-harmonic oscillation. Hence, (2.15) can be re-expressed as

$$\frac{2\pi\tau}{T} \ll 1 \tag{2.24}$$

The above implies that if the transit time τ needed to traverse the object of length L is much small than the period of oscillation of the electromagnetic field, then static theory can be used.

The finite speed of light gives rise to delay or retardation of electromagnetic signal when it propagates through space. When this retardation effect can be ignored, then static theory can be used. In other words, if the speed of light had been infinite, then there would be no retardation effect, and static theory could always be used. Alternatively, the infinite speed of light will give rise to infinite wavelength, and criterion (2.16) will always be satisfied, and static theory prevails.

2.1 Quasi-Electromagnetic Theory

In closing, we would like to make one more remark. The right-hand side of (2.11), which is Faraday's law, is essential for capturing the physical mechanism of an inductor and flux linkage. And yet, if we drop it, there will be no inductor in this world. To understand this dilemma, let us rewrite (2.11) in integral form, namely,

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -j\omega\mu_0 \frac{L}{\eta_0} \iint_S d\mathbf{S} \cdot \mathbf{H} \tag{2.25}$$

In the inductor, the right-hand side has been amplified by multiple turns, effectively increasing S, the flux linkage area. Or one can think of an inductor as having a much longer effective length L_{eff} when untwined so as to compensate for decreasing frequency ω. Hence, the importance of flux linkage or the inductor in circuit theory is not diminished unless $\omega = 0$.

By the same token, displacement current can be enlarged by using capacitors. In this case, even when no electric current $\mathbf{J}$ flows through the capacitor, displacement current flows and the generalized Ampere's law becomes

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = j\omega\varepsilon\eta_0 L \iint_S d\mathbf{S} \cdot \mathbf{E}' \tag{2.26}$$

The displacement in a capacitor cannot be ignored unless $\omega = 0$. Therefore, when $\omega \neq 0$, or in quasi-static case, inductors and capacitors in circuit theory are important as we shall study next.